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An Application of STWS Technique in Solving Stiff Non-linear System: 'High Irradiance Responses' (HIRES) of Photomorphogenesis

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Abstract

This paper illustrates an application of the Single Term Walsh Series (STWS) technique in solving stiff non-linear system: 'High Irradiance RESponses' (HIRES) of Photomorphogenesis from plant physiology. The chemical reaction scheme of HIRES problem has been modelled into system of stiff non-linear differential equations. This stiff system has been solved using the STWS technique. The STWS solutions are compared with the results obtained by the well-known solvers, namely, VODE and RADAU5. The applicability of the STWS technique has been tested.

Key Words: HIRES; STWS; WF; BPF; Stiff; Non-linear

Introduction

Many physical systems such as nuclear reactors and laser oscillators give rise to stiff non-linear ordinary differential equations (ODEs) in which the magnitudes of the eigenvalues vary greatly. Methods not designed for stiff problems are ineffective on intervals where the solution changes slowly because they use time steps small enough to resolve the fastest possible change. Stiff problems typically arise in chemical kinetics, nuclear reactor theory, control theory, biochemistry, climatology, electronics, fluid dynamics, etc. The numerical integration of a stiff system has attracted many researchers. This paper deals with the STWS technique to determine the numerical solution of stiff non-linear system.

The STWS technique was introduced by Rao et al. [1] using Walsh Functions (WFs). This is based on an approach called "single segment approximation" which avoids operational matrices of large size and maximizes the reduction in the computational effort. This method provides discrete solutions of problems, to any length of time, in an easy manner. Many researchers have made use of this STWS technique to solve different systems. Palanisamy [2] have used the STWS approach to the analysis and optimal control of linear time invariant and time varying systems. Subbayyan et al. [3] have used the STWS method for the optimal control of singularly perturbed linear systems. Palanisamy and Arunachalam [4] have presented the STWS method for the analysis of bilinear systems. Balachandran and Murugesan [5] have applied STWS technique to the analysis of linear and non-linear singular systems. Murugesan et al. [6] have obtained the STWS solutions of the second order multi-variable linear system.

Sepehrian and Razzaghi [7] have proposed STWS method to find the solution of time-varying singular non-linear systems. Dhayabaran et al. [8] have obtained the numerical solution of Robot Arm Model using STWS method and extended Runge-Kutta method based on Heronian Mean (RKHeM). Emimal et al. [9] have proposed a generalised STWS technique to determine discrete solutions for the systems of IVPs of any higher order '*n*' with '*p*' variables. Dhayabaran et al. [10] have applied this generalized STWS technique to higher order time-varying singular systems.

This paper illustrates an application of the STWS technique in solving stiff non-linear system: 'High Irradiance RESponses' (HIRES) of Photomorphogenesis from the field of plant physiology. The chemical reaction scheme of HIRES problem has been modelled into system of eight stiff non-linear differential equations. This stiff system has been solved using the STWS technique. The STWS solutions are compared with the results obtained by the well-known solvers, namely, VODE by Brown et al. [11] which is based upon linear multistep methods and RADAU5 by Hairer and Wanner [12] which is based on implicit RK method.

STWS Technique

It is well known that a function, which is periodic, may be expanded into a Fourier series or Power series. In a similar manner, a function f(t), which is integrable in [0,1) may be approximated by using WFs as

$$f(t) = \sum_{i=0}^{\infty} f_i \, \psi_0(t)$$

where $\psi_0(t)$ is the i[th] WF and f_i is the corresponding coefficient. In practice, while approximating a function, only the first 'm' components are considered. If the coefficients of the WFs are concisely written as 'm' vectors, then

$F = (f_0, f_1, \ldots, f_{m-1})^T$ and

$\psi(t) = [\, \psi_0(t), \; \psi_1(t), \ldots, \; \psi_{m-1}(t) \,]^T$

where $m = 2^k$, k is an integer and T denotes transpose. Then the above function f(t) becomes

$f(t) \approx F^T \psi(t)$.

The coefficients f_i are chosen to minimise the mean integral square error

$$\varepsilon = \int_0^1 [\, f(t) - F^T \psi(t)]^2 \, dt.$$

The coefficients are given by

$$f_i = \int_0^1 f(t)\psi_i(t)dt.$$

It has been proved by Chen and Hsiao [13] that

$$\int_0^t f(t)dt \approx F^T E \psi(t).$$

The constant square matrix E is called the operational matrix for integration in F and is given by

$$E_{(m\times m)} = \begin{pmatrix} E_{(m/2\times m/2)} & -\frac{1}{2m} I_{(m/2\times m/2)} \\ \frac{1}{2m} I_{(m/2\times m/2)} & O_{(m/2\times m/2)} \end{pmatrix}$$

with $E_{(2\times 2)} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix}$ and $E_{(1\times 1)} = \frac{1}{2}.$

A function x(t) may be expanded into a STWS in the normalised interval $s \in [0, 1)$ (which corresponds to $t \in [0, 1/m)$ by defining $\tau = mt$) as

$x(\tau) = B_1 \psi_0(\tau).$

Here B_1 is the block pulse function (BPF) value of x(t) in $t \in [0, 1/m)$. The operational matrix E for integration becomes a scalar, since $E_{(1\times 1)} = \frac{1}{2}$.

Consider the integral $\int_0^1 \dot{x}(\tau) d\tau$.

Express $\dot{x}(\tau)$ in STWS in the first normalized interval $\tau \in [0, 1)$ as

$$\dot{x}(\tau) = C_1 \psi_0(\tau) \qquad (2.1)$$

Here C_1 is the BPF value of the rate vector $\dot{x}(t)$.

Integrating Eqn. (2.1) with the operational matrix E = $\frac{1}{2}$,

$$B_1 \psi_0(\tau) = \frac{1}{2} C_1 \psi_0(\tau) + x(0) \qquad (2.2)$$

According to the definition of Sannuti [14], the BPF value of $\dot{x}(t)$ in the first interval is given by

$$C_1 = m \int_0^{1/m} \dot{x}(t)dt, \quad t \in [0, 1/m). \qquad (2.3)$$

When Eqn. (2.3) is normalized with t=s/m

$$C_1 = \int_0^1 \dot{x}(\tau)d\tau = x(1) - x(0), \qquad \tau \in [0,1)$$

$x(1) = C_1 + x(0)$.

For any interval k,

$$x(k) = C_k + x(k-1). \qquad (2.4)$$

Eqn. (2.4) gives an important way to get discrete time values of x(t) and to continue the STWS solution to the next normalized interval.

STWS Technique for Solving Non-linear Systems

Consider a first order non-linear system of the following form:

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(0) = x_0, \qquad (3.1)$$

where the non-linear function $f \in R^n$, the state $x(t) \in R^n$, and the control $u(t) \in R^q$. With the STWS approach, the given function is expanded in the normalized interval $\tau \in [0, 1)$, which corresponds to $t \in [0,1/m)$ by defining $\tau = mt$, *m* being any integer. Normalizing Eqn. (3.1), we get

$$m\dot{x}(\tau) = f(\tau, x(\tau), u(\tau)), \quad x(0) = x_0 \qquad (3.2)$$

Let $\dot{x}(\tau)$ and $x(\tau)$ be expanded by STWS series in the k^{th} interval as

$$\dot{x}(\tau) = V^{(k)} \psi_0(\tau) \text{ and } x(\tau) = X^{(k)} \psi_0(\tau). \qquad (3.3)$$

Integrating Eqn. (3.3) with the operational matrix E = $\frac{1}{2}$,

we get

$$X^{(k)} = \frac{1}{2} V^{(k)} + x(k-1) \text{ and } x(k) = V^{(k)} + x(k-1). \qquad (3.4)$$

Therefore, $x(\tau) = \left(\frac{1}{2}V^{(k)} + x(k-1)\right)\psi_0(\tau) \qquad (3.5)$

To solve Eqn. (3.2), we first substitute Eqn. (3.5) in $f(\tau, x(\tau), u(\tau))$. Then we express the resulting equation by STWS as

$$f\left(\tau, \left(\frac{1}{2}V^{(k)} + x(k-1)\right)\psi_0(\tau), u(\tau)\right) = F^{(k)}\psi_0(\tau) \qquad (3.6)$$

Using the Eqns. (3.2), (3.3), (3.5), and (3.6), we get

$$mV^{(k)} = F^{(k)}. \qquad (3.7)$$

By solving Eqn. (3.7), the components of $V^{(k)}$ can be obtained. Then, by substituting $V^{(k)}$ in Eqn. (3.4), we obtain discrete approximations of the state x(t).

Description of HIRES Problem

The HIRES problem was proposed by Schäfer [15] in 1975. The parallel-IVP-algorithm group of Centre for Mathematics and Computer Science have contributed this problem to the test set [16]. This problem describes how light is involved in morphogenesis. It explains the `High Irradiance Responses' of photomorphogenesis on the basis of phytochrome, by means of a chemical reaction involving eight reactants.

The reaction scheme for the HIRES problem is given in Fig. 4.1. P_r and P_{fr} refer to the red and far-red absorbing form of phytochrome, respectively. They can be bound by two receptors X and X', partially influenced by the enzyme E. The values of the parameters were taken as

$$k_1 = 1.71 \qquad k_6 = 8.32$$
$$k_2 = 0.43 \qquad k_+ = 280$$
$$k_3 = 8.32 \qquad k_- = 0.69$$
$$k_4 = 0.69 \qquad k^* = 0.69$$
$$k_5 = 0.035 \qquad o_{k_s} = 0.0007$$

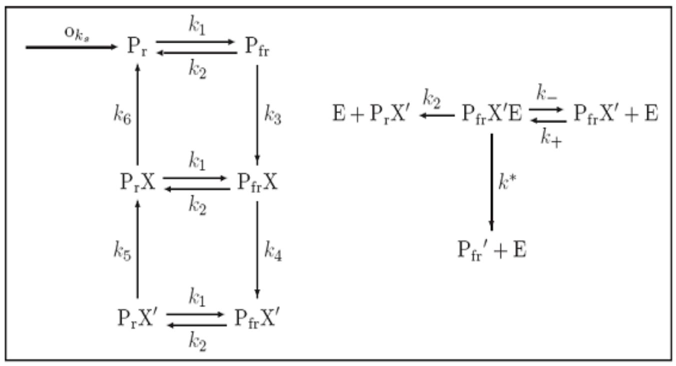


Fig. 4.1

The chemical process is started at $t_0 = 0$ by mixing one mol of P_r with 0.0057 mol of E. The integration process is continued during 321.8122 seconds. This value and the parameter values were chosen by Hairer and Wanner [17].

The reaction scheme for the HIRES problem can be described by a set of differential equations as follows:

$$\mathrm{d}[\mathrm{P}_r]/\mathrm{d}t = -k_1[\mathrm{P}_r] + k_2[\mathrm{P}_{fr}] + k_6[\mathrm{P}_r\mathrm{X}] + \mathrm{o}_{k_s}$$

$$\mathrm{d}[\mathrm{P}_{fr}]/\mathrm{d}t = k_1[\mathrm{P}_r] - (k_2+k_3)[\mathrm{P}_{fr}]$$

$$\mathrm{d}[\mathrm{P}_r X]/\mathrm{d}t = -(k_1+k_6)[\mathrm{P}_r X] + k_2[\mathrm{P}_{fr} X] + k_5[\mathrm{P}_r \mathrm{X}']$$

$$\mathrm{d}[\mathrm{P}_{fr} X]/\mathrm{d}t = k_3[\mathrm{P}_{fr}] + k_1[\mathrm{P}_r X] - (k_2+k_4)[\mathrm{P}_{fr} X]$$

$$\mathrm{d}[\mathrm{P}_r X']/\mathrm{d}t = -(k_1+k_5)[\mathrm{P}_r X'] + k_2[\mathrm{P}_{fr} X'] + k_2[\mathrm{P}_{fr}\mathrm{X}'\mathrm{E}]$$

$$\mathrm{d}[\mathrm{P}_{fr} X']/\mathrm{d}t = k_4[\mathrm{P}_{fr} X] + k_1[\mathrm{P}_r X'] - k_2[\mathrm{P}_{fr} X'] + k_-[\mathrm{P}_{fr} X' E] - k_+[\mathrm{P}_{fr} X'][E]$$

$$\mathrm{d}[\mathrm{P}_{fr} X' E]/\mathrm{d}t = -(k_2+k_-+k^*)[\mathrm{P}_{fr} X' E] + k_+[\mathrm{P}_{fr} X'][E]$$

$$\mathrm{d}[E]/\mathrm{d}t = (k_2+k_-+k^*)[\mathrm{P}_{fr} X' E] - k_+[\mathrm{P}_{fr} X'][E].$$

Here the square brackets '[ ]' denote concentrations. Identifying the concentrations of P_r, P_{fr}, P_rX, $P_{fr}X$, $P_r X'$, $P_{fr} X'$, $P_{fr} X' E$ and E with $y_i, i \in \{1, 2, ..., 8\}$, respectively, the above reaction scheme for HIRES problem can be transformed into the system of stiff non-linear differential equations as follows:

$$\frac{dy}{dt} = \mathrm{f(y)}, \qquad \mathrm{y(0)} = \mathrm{y}_0, with$$
$$y \in R^8, 0 \le t \le 321.8122.$$

The function f is defined by

$$f(y) = \begin{pmatrix} -1.71y_1 + 0.43y_2 + 8.32y_3 + 0.0007 \\ 1.71y_1 - 8.75y_2 \\ -10.03y_3 + 0.43y_4 + 0.035y_5 \\ 8.32y_2 + 1.71y_3 - 1.12y_4 \\ -1.745y_5 + 0.43y_6 + 0.43y_7 \\ -280y_6y_8 + 0.69y_4 + 1.71y_5 - 0.43y_6 + 0.69y_7 \\ 280y_6y_8 - 1.81y_7 \\ -280y_6y_8 + 1.81y_7 \end{pmatrix}$$

The initial vector y_0 is given by $(1, 0, 0, 0, 0, 0, 0, 0.0057)^T$.

Applicability of STWS Technique in Solving HIRES Problem

The discrete solutions of HIRES problem can be determined using the STWS technique discussed in Section 3. The STWS solutions at the end of the integration interval are compared with the results obtained by the well-known solvers VODE and RADAU5. These results have been shown in Table 5.1. The solution graphs by the solver RADAU5 are given in Fig. 5.2(a) – Fig. 5.2(h) and the solution graphs by the STWS technique are given in Fig. 5.3(a)–Fig.5.3(h).

Table 5.1: Solutions of HIRES Problem

Variables	STWS	VODE	RADAU5
y_1	0.0007382	0.0007405	0.0007371
y_2	0.0001445	0.0001449	0.0001442
y_3	0.0000591	0.0000595	0.0000589
y_4	0.0011776	0.0011821	0.0011757
y_5	0.0024179	0.0024836	0.0023863
y_6	0.0063381	0.0064948	0.0062390
y_7	0.0028720	0.0029543	0.0028500
y_8	0.0027926	0.0027457	0.0028500

Fig. 5.2(a)

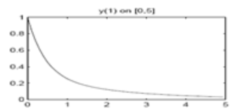


Fig. 5.3(a) y_1 on [0,5]

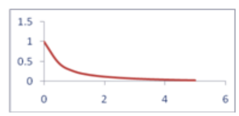


Fig. 5.2(b)

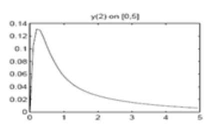


Fig. 5.3(b) y_2 on [0,5]

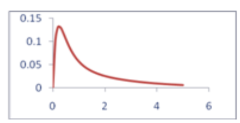


Fig. 5.2(c)

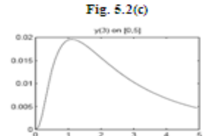


Fig. 5.3(c) y_3 on [0,5]

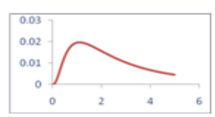


Fig. 5.2(d)

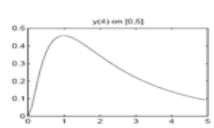


Fig. 5.3(d) y_4 on [0,5]

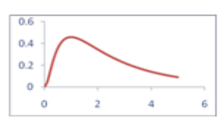


Fig. 5.2(e)

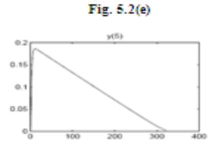


Fig. 5.3(e) y_5

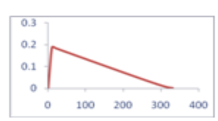


Fig. 5.2(f)

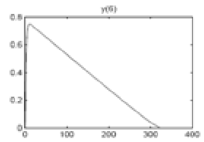


Fig. 5.3(f) y_6

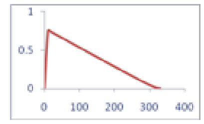


Fig. 5.2(g)

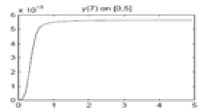


Fig. 5.3(g) y_7 on [0, 5]

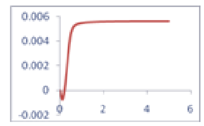


Fig. 5.2(h)

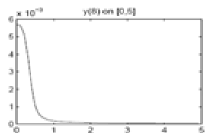
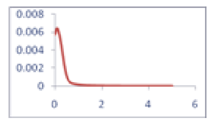


Fig. 5.3(h) y_8 on [0,5]



To demonstrate the applicability of the STWS technique, the discrete solutions of this problem have been determined in the interval [0, 5] by varying the values of 'm' (involved in the STWS technique), for example, m = 100, 200, 300, 400, 500, 600. The absolute errors between the STWS solutions of y_1, y_2, . . . , y_8 in [0, 5] at two consecutive values of 'm', say m_1 and m_2, have been determined which may be denoted as $Err_{m_1}^{m_2}$.

But in this paper, as an illustration, the error results for the variables y_1 and y_4 have been given in Tables 5.2 - 5.5. Further, the maximum errors for the variables y_1, y_2, . . . , y_8 in [0, 5] have been determined and shown in Table 5.6.

Table 5.2

Time t	STWS Solution of y_1					
	m = 100	m = 200	m = 300	m = 400	m = 500	m = 600
0	1	1	1	1	1	1
1	0.255485	0.255491	0.255492	0.255492	0.255492	0.255493
2	0.123519	0.123519	0.123520	0.123520	0.123520	0.123520
3	0.07470	0.074697	0.074697	0.074697	0.074697	0.074697
4	0.047824	0.04782	0.047823	0.047823	0.047823	0.047823
5	0.031652	0.031652	0.031652	0.031652	0.031652	0.031652

Table 5.3

Time t	Absolute Error in y_1				
	Err_{100}^{200}	Err_{200}^{300}	Err_{300}^{400}	Err_{400}^{500}	Err_{500}^{600}
0	0	0	0	0	0
1	5.5E-06	1.0E-06	4.0E-07	2.0E-07	1.0E-07
2	8.0E-07	2.0E-07	1.0E-07	0	0
3	3.0E-07	0	0	1.0E-07	0
4	6.0E-07	0	1.0E-07	0	0
5	7.0E-07	1.0E-07	0	0	0

Table 5.4

Time t	STWS Solution of y_4					
	m = 100	m = 200	m = 300	m = 400	m = 500	m = 600
0	0	0	0	0	0	0
1	0.458531	0.458522	0.458521	0.458520	0.458520	0.458520
2	0.344056	0.344054	0.344054	0.344053	0.344053	0.344053
3	0.220462	0.220462	0.220462	0.220462	0.220462	0.220462
4	0.139693	0.139692	0.139692	0.139692	0.139692	0.139692
5	0.089744	0.089743	0.089743	0.089743	0.089743	0.089743

Table 5.5

Time t	Absolute Error in y_4				
	Err_{100}^{200}	Err_{200}^{300}	Err_{300}^{400}	Err_{400}^{500}	Err_{500}^{600}
0	0	0	0	0	0
1	8.8E-06	1.6E-06	6E-07	3E-07	1E-07
2	1.8E-06	2E-07	1E-07	0	1E-07
3	7E-07	1E-07	0	0	0
4	7E-07	0	1E-07	0	0
5	1E-06	1E-07	0	0	0

Table 5.6

Variables	Maximum Error in [0, 5]				
	Err_{100}^{200}	Err_{200}^{300}	Err_{300}^{400}	Err_{400}^{500}	Err_{500}^{600}
y_1	5.50E-06	1E-06	4E-07	2E-07	1E-07
y_2	1.20E-06	3E-07	1E-07	0	0
y_3	6.00E-07	1E-07	0	0	0
y_4	8.80E-06	1.6E-06	6E-07	3E-07	1E-07
y_5	2.85E-05	1E-07	0	0	0
y_6	1.19E-04	4E-07	1E-07	1E-07	0
y_7	1.00E-07	0	0	0	0
y_8	1.00E-07	0	0	0	0

Conclusion

In this paper, the chemical reaction scheme of HIRES problem has been transformed into a system of eight stiff non-linear differential equations. The STWS technique has been applied to solve this problem. From the graphs given in Fig. 5.3(a) – (h), it is observed that after about 320 seconds, the concentrations of the species are very small, which indicates that almost all phytochrome has been transformed to P_{fr}'. The amount of P_{fr}' can be derived from the conservation of the total amount of phytochrome.

From Table 5.1, it is observed that the STWS solutions agree well with that of the solvers VODE and RADAU5. In order to ascertain the applicability of the STWS technique, a thorough study has been undertaken by determining the STWS solutions for the decision variables involved in HIRES problem in [0, 5] by varying the values of m = 100, 200, 300, 400, 500, 600. From Table 5.6, it is observed that the error decreases as the value of 'm' increases.

Hence, it is concluded that the proposed STWS technique is stable and very much applicable in solving the real-world problems like HIRES problem and similar realistic problems existing in various fields of science and engineering.

References

[1] G.P. Rao, K.R. Palanisamy, T. Srinivasan, Extension of computation beyond the limit of normal interval in Walsh Series analysis of dynamical systems, *IEEE Trans. Autom. Control 25* (1980) 317 – 319.

[2] K. R. Palanisamy, Analysis and optimal control of linear systems via single-term Walsh series approach, *Int. J. Systems Sci.* 12 (1981) 443 – 454.

[3] R. Subbayyan, K. Muhammad Zakariah, K. R. Palanisamy, Optimal Control of Singularly Perturbed Linear Systems: Single Term Walsh Series Approach, *Int. J. Systems Sci.* 13 (1982) 1339 – 1343.

[4] K. R. Palanisamy, V. P. Arunachalam, Analysis of Bilinear Systems via Single-Term Walsh Series, *Int. J. Control* 41 (1985) 541 – 547.

[5] K. Balachandran, K. Murugesan, Analysis of non-linear singular systems via STWS method, *Int. J. Comp. Math.* 36 (1990) 9 – 12.

[6] K. Murugesan, D. P. Dhayabaran, David J. Evans, Analysis of second order multivariable linear system using single-term Walsh series technique and Runge Kutta method, *Int. J. Comp. Math.* 72 (1999) 367 – 374.

[7] B. Sepehrian, M. Razzaghi, Solution of Time-varying Singular Nonlinear Systems by Single Term Walsh

Series, *Mathematical Problems in Engineering* 2003:3 (2003) 129 – 136.

[8] D. P. Dhayabaran, E. C. Henry Amirtharaj, K. Murugesan, Numerical Solution of Robot Arm Model using STWS and RKHeM Techniques, *Proc. First International Conference on Computational Methods,* Springer Netherlands (2006) 1695 – 1699.

A. Emimal Kanaga Pushpam, D. P. Dhayabaran, E. C. Henry Amirtharaj, Numerical solution of higher order systems of IVPs using generalized STWS technique, *Applied Mathematics and Computation* 180 (2006) 200 – 205.

[9] D. P. Dhayabaran, A. Emimal Kanaga Pushpam, E. C. Henry Amirtharaj, Generalized STWS Technique for Higher Order Time-varying Singular Systems, *Int. J. Comput. Math*., 84 (2007) 395 – 402.

[10] P. N. Brown, G. D. Byrne, A.C. Hindmarsh, Vode: A variable coefficient ode solver, *SIAM J. Sci. Stat. Comput.,* 10 (1989) 1038 – 1051.

[11] E. Hairer, G. Wanner, RADAU5, CWI Publication, 1996. Also available at ftp://ftp.unige.ch/pub/doc /math/stiff/radau5.f.

[12] C. F. Chen, C. H. Hsiao, A State Space Approach to Walsh Series Solution of Linear Systems, *Int. J. Systems Sci.*, 6 (1975) 833 - 858.

[13] P. Sannuti, Analysis and Synthesis of Dynamic Systems via Block-Pulse Functions, *Proc. IEE*, 124 (1977) 569 – 571.

[14] E. Sch $\ddot{a}$ fer, A New Approach to explain the `High Irradiance Responses' of Photomorphogenesis on the Basis of Phytochrome, *J. Math. Biology*, 2 (1975) 41 - 56.

[15] W. M. Lioen, J. J. B. de Swart, Test Set for Initial Value Problem Solvers, CWI Publication, 1999. Also available at URL http://www.cwi.nl/cwi/projects/IVP testset.html.

[16] E. Hairer, G. Wanner, Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems, Springer-Verlag,1996.